Published by Institute of Physics Publishing for SISSA

Received: *June 9, 2008*
Accepted: *July 16, 2008*
Published: *July 29, 2008*

# Description of D-branes invariant under the Poisson-Lie T-plurality

**Ladislav Hlavatý and Libor Šnobl***

*Faculty of Nuclear Sciences and Physical Engineering,*
*Czech Technical University in Prague,*
*Břehová 7, 115 19 Prague 1, Czech Republic*
*E-mail:* Ladislav.Hlavaty@fjfi.cvut.cz, Libor.Snobl@fjfi.cvut.cz

Abstract: We write the conditions for open strings with charged endpoints in the language of gluing matrices. We identify constraints imposed on the gluing matrices that are essential in this setup and investigate the question of their invariance under the Poisson-Lie T-plurality transformations. We show that the chosen set of constraints is equivalent to the statement that the lifts of D-branes into the Drinfel'd double are right cosets with respect to a maximally isotropic subgroup and therefore it is invariant under the Poisson-Lie T-plurality transformations.

Keywords: D-branes, String Duality.

*This work was supported by the project No. 202/06/1480 of the Grant Agency of the Czech Republic and by the research plans LC527 and MSM6840770039 of the Ministry of Education of the Czech Republic.

## Contents



JHEP07(2008)122

## 1. Introduction

In our previous paper [1], the transformation of worldsheet boundary conditions for non-linear sigma models under the Poisson-Lie T-plurality [2, 3] was investigated and a formula for the transformation of gluing matrices was presented there. Boundary conditions were formulated in terms of a so-called gluing matrix that was subjected to a set of constraints originally formulated for supersymmetric models in [4, 5]. Abelian T-duality of such models (and also of their purely bosonic analogues) was studied in [6] and later also partially extended to Poisson-Lie T-duality context in [7]. Unfortunately, we have shown in [1] that some of the constraints are not preserved under the Poisson-Lie transformation (even in the simplest non-Abelian T-duality context).

In this paper we present a restricted set of constraints for the gluing matrix that does not disqualify the interpretation of corresponding boundary condition in terms of D-branes and simultaneously preserves its validity under the Poisson-Lie transformations. It means that well defined D-branes formulated in this way transform into well defined D-branes again under the Poisson-Lie T-plurality.

The existence of such description was to be expected because there exists a different, geometric formulation of the same problem based on the geometry of D-branes lifted into the Drinfel'd double by C. Klimčík and P. Ševera in [8, 9]. The open problem was how to express their formulation in the language of gluing matrices, i.e., how their boundary conditions manifest themselves on the level of original $\sigma$-models.

The paper is structured as follows. Firstly, we review and modify the formulation of boundary conditions in terms of gluing matrices (or operators) $\mathcal{R}$. Secondly, we recall some of the basic properties of Poisson-Lie T-duality and plurality and how the gluing matrices transform. Thirdly, we demonstrate a few examples we have used in the search for consistency constraints on $\mathcal{R}$ preserved under Poisson-Lie transformations. Next, we rewrite the constraints on $\mathcal{R}$ in an equivalent form suitable for further computations (i.e. without projectors). Finally, we lift the D-branes into the Drinfel'd double, study how the boundary conditions manifest themselves there, demonstrate the connection with the description in [8] and show the invariance of our constraints.

## 2. Boundary conditions and D-branes

We investigate the boundary conditions for equations of motion of nonlinear sigma models given by the action[1]

$$S_{\mathcal{F}}[\phi] = \int_{\Sigma} d^2x\, \partial_-\phi^\mu \mathcal{F}_{\mu\nu}(\phi)\partial_+\phi^\nu = \int_{\Sigma} d^2x\, \partial_-\phi \cdot \mathcal{F}(\phi) \cdot \partial_+\phi^t \tag{2.1}$$

where $\mathcal{F}$ is a tensor field on a Lie group $G$ and the functions $\phi^\mu : \Sigma \subset \mathbb{R}^2 \to \mathbb{R}$, $\mu = 1, 2, \ldots, \dim G$ are obtained by the composition $\phi^\mu = y^\mu \circ g$ of a map $g : \Sigma \to G$ and components of a coordinate map $y$ of a neighborhood $U_g$ of an element $g(x_+, x_-) \in G$. For the purpose of this paper we shall assume that the worldsheet $\Sigma$ has a topology of a strip infinite in $\tau \equiv x_+ + x_-$ direction, $\Sigma = \mathbb{R} \times \langle 0, \pi\rangle$ and $x_+, x_-$ are light-cone coordinates on $\Sigma$.

We impose the boundary conditions for open strings in the form of the gluing operator $\mathcal{R}$ relating left and right derivatives of the field $g : \Sigma \to G$ on the boundary of $\Sigma$,

$$\partial_- g|_{\sigma=0,\pi} = \mathcal{R}\, \partial_+ g|_{\sigma=0,\pi}, \qquad \sigma \equiv x_+ - x_-. \tag{2.2}$$

We denote the matrices corresponding to the operator $\mathcal{R}$ on $T_gG$ in the bases of coordinate derivatives $\partial_{y^\mu}$ as $R$, e.g.,[2]

$$\partial_-\phi|_{\sigma=0,\pi} = \partial_+\phi \cdot R|_{\sigma=0,\pi}. \tag{2.3}$$

The explicit form of the operator $\mathcal{R}$ in principle yields the embedding of a brane in the target space which is in this case the Lie group $G$.

When varying the action (2.1) we shall impose vanishing of boundary terms

$$\delta\phi \cdot (\mathcal{G} \cdot \partial_\sigma \phi^t + \mathcal{H} \cdot \partial_\tau \phi^t)|_{\sigma=0,\pi} = 0, \tag{2.4}$$

[1] We use a bit unusual notation that $\partial_\pm\phi^\mu$ form row vectors of the derivatives of $\phi$, therefore matrices of operators in our notation may differ by a transposition from expressions in other papers. The dot denotes matrix multiplication, $t$ denotes transposition, $X^{-t} \equiv (X^t)^{-1}$.

[2] Similarly we shall distinguish operators from their matrices by the calligraphic script used. This does not apply to tensorial expressions $\mathcal{F}, \mathcal{G}, \mathcal{H}$.



JHEP07(2008)122

where $\mathcal{G}$ and $\mathcal{H}$ are symmetric and antisymmetric part of the tensor field $\mathcal{F}$. We shall assume that the ends of an open string move along a D-brane — submanifold $\mathcal{D} \subset G$ – so that both $\delta\phi|_{\sigma=0,\pi} \in T_g\mathcal{D}$ and $\partial_\tau\phi|_{\sigma=0,\pi} \in T_g\mathcal{D}$. Let $\mathcal{N}$ be a projector $T_gG \to T_g\mathcal{D}$ so that

$$\delta\phi|_{\sigma=0,\pi} = \mathcal{N}(\delta\phi|_{\sigma=0,\pi}), \quad \partial_\tau\phi|_{\sigma=0,\pi} = \mathcal{N}(\partial_\tau\phi|_{\sigma=0,\pi}). \tag{2.5}$$

From eqs. (2.3) and (2.5) we may express the defining properties of $\mathcal{N}$ as

$$\mathcal{N} \circ (\mathcal{R} + id) = (\mathcal{R} + id),\ \mathcal{N}^2 = \mathcal{N},\ \mathrm{Ran}\,\mathcal{N} = \mathrm{Ran}\,(\mathcal{R} + id),$$

i.e.

$$(R + \mathbf{1}) \cdot N = R + \mathbf{1},\ N^2 = N,\ \mathrm{rank}\,N = \mathrm{rank}\,(R + \mathbf{1}). \tag{2.6}$$

We should stress that these properties do not specify the projector $\mathcal{N}$ uniquely since its kernel is not determined. As it will become clear later on, we may consider all such projectors equivalent for any sensible use in physics.

We can rewrite the equation (2.4) as

$$\delta\phi \cdot N \cdot (\mathcal{F} \cdot \partial_+\phi^t - \mathcal{F}^t \cdot \partial_-\phi^t)|_{\sigma=0,\pi} = 0, \tag{2.7}$$

which after the use of eq. (2.3) becomes

$$\delta\phi \cdot N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) \cdot \partial_+\phi^t|_{\sigma=0,\pi} = 0. \tag{2.8}$$

Because $\delta\phi \cdot N$ and $\partial_+\phi^t$ are not further restricted, we find

$$N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0. \tag{2.9}$$

Besides that there are conditions for $N$ and $R$

$$\begin{aligned} N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}^{\ \rho} &= 0, \\ R \cdot \mathcal{G} \cdot R^t &= \mathcal{G} \end{aligned} \tag{2.10}$$

that follow from the condition that the projectors $\mathcal{N}$ in all points of $G$ define integrable distribution and that the stress tensor of the action vanishes on the boundary (see e.g. [1, 6, 10]).

In our previous paper [1], we have used the formulation first presented in [5], i.e. we have defined D-branes by virtue of Dirichlet projector $\mathcal{Q}$ that projects tangent vectors in a point of $G$ onto the space normal to the D-brane going through this point and the normal space was identified with the eigenspace of $\mathcal{R}$ with the eigenvalue $-1$, i.e.,

$$Q^2 = Q,\ Q \cdot R = -Q. \tag{2.11}$$

The Neumann projector $\mathcal{N}$, which projects onto the tangent space of the brane was then defined as complementary to $\mathcal{Q}$, i.e.

$$N := \mathbf{1} - Q.$$



JHEP07(2008)122

The eq. (2.11) is then equivalent to

$$N^2 = N,\ N \cdot (R + \mathbf{1}) = R + \mathbf{1}.$$

In order to get an agreement with eq. (2.6) we had to assume that the geometrical and algebraic multiplicities of the eigenvalue $-1$ are equal. This gave another condition that relates $R$ and $Q$

$$Q \cdot R = R \cdot Q \tag{2.12}$$

so that we got the following set of conditions (equivalent to those in [5])

$$Q^2 = Q,\ Q \cdot R = -Q,\ \operatorname{rank} Q = \dim\ker (R + \mathbf{1}) \tag{2.13}$$
$$Q \cdot R = R \cdot Q, \tag{2.14}$$
$$N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho = 0, \tag{2.15}$$
$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{2.16}$$
$$N \cdot (\mathcal{F} - \mathcal{F}^t \cdot R^t) = 0. \tag{2.17}$$

We found in our previous work that the constraints for a consistent gluing operator $\mathcal{R}$ derived above are not in general preserved under the Poisson-Lie transformations (see section 5.2, case (100) in [1]).

The situation improved a bit when we admitted that the endpoints of the string are electrically charged so that the action must be modified by boundary terms. Such an extension in the context of Poisson-Lie T-duality of open strings was already introduced in [8], in the gluing matrix language was firstly mentioned in [5]. We have

$$S_{\mathcal{F}}[\phi] \to S_{\mathcal{F}}[\phi] + S_{\text{boundary}}[\phi] \tag{2.18}$$

where

$$S_{\text{boundary}}[\phi] = q_0 \int_{\sigma=0} A_\mu \frac{\partial \phi^\mu}{\partial \tau} \mathrm{d}\tau - q_0 \int_{\sigma=\pi} A_\mu \frac{\partial \phi^\mu}{\partial \tau} \mathrm{d}\tau \tag{2.19}$$

corresponds to electrical charges $q_0, -q_0$ associated with the two endpoints of the string interacting with electric field(s) present on the respective D-branes. The condition (2.9) is then modified to the form [1]

$$N \cdot \big((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t\big) = 0, \tag{2.20}$$

where in local coordinates adapted to the brane[3] we have

$$\Delta_{\mu\nu} = \frac{1}{2} \left( \frac{\partial A_\nu}{\partial y^\mu} - \frac{\partial A_\mu}{\partial y^\nu} \right), \tag{2.21}$$

$\mu, \nu \leq \dim(\text{brane})$ (the remaining components of $\Delta$ do not contribute to the eq. (2.20)). For computational simplicity we assume that $\Delta$ can be smoothly extended into a neighborhood of the brane. Because the values of $\Delta$ are physically relevant only along the D-brane we

[3] i.e., $\frac{\partial}{\partial y^\mu}$, $\mu = 1, \ldots, \dim(\text{brane})$ are tangential to the brane and the remaining vectors $\frac{\partial}{\partial y^\kappa}$, $\kappa > \dim(\text{brane})$ are transversal.



JHEP07(2008)122

may impose a supplementary restriction on $\Delta$ that fixes its extension into the transversal directions

$$\Delta = N \cdot \Delta \cdot N^t. \tag{2.22}$$

The exactness of $\Delta$ along the brane (2.21) is locally equivalent to its closeness written in arbitrary coordinates as

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu}\Delta_{\rho\sigma]} = 0. \tag{2.23}$$

Unfortunately, neither this generalized formulation of D-branes defined by the gluing operator and interaction with the charges is preserved under the Poisson-Lie T-plurality or Poisson-Lie T-duality in the sense that there are cases when the set of conditions (2.13)–(2.16), (2.20) and (2.23) holds for a $\sigma$-model with boundary conditions given by $\mathcal{R}$ but not for a model and boundary conditions obtained by the Poisson-Lie transformation (See section 5.2, case (101) in [1]).

This problem forces us to reconsider the necessity of conditions (2.13)–(2.16). Namely, motivated by explicit examples in [1] we revisit the condition (2.14). If this condition holds (as is always the case when $\mathcal{G}$ is positive/negative definite but not in general) then there is a canonical choice of the projector $\mathcal{N}$, namely, such that $\mathcal{N}$ is an orthogonal projector with respect to the metric $\mathcal{G}$. On the other hand, if the condition (2.14) does not hold, one cannot choose the projector $\mathcal{N}$ uniquely and also it is not possible to find the so-called adapted coordinates [5], i.e. the boundary conditions cannot be split into Dirichlet and (generalized) Neumann directions. Although such boundary conditions may appear strange, we don't see any reason why they should be a priori excluded from consideration.

Moreover, we shall prove that if we relax the condition (2.14) and reformulate the other ones in such a way that the $\sigma$-model with boundary conditions is given by $(\mathcal{F}, \mathcal{R}, \Delta)$ satisfying

$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{2.24}$$

$$(R + \mathbf{1}) \cdot N = (R + \mathbf{1}), \quad N^2 = N, \quad \mathrm{rank}\, N = \mathrm{rank}\,(R + \mathbf{1}) \tag{2.25}$$

$$N_\kappa{}^\mu N_\lambda{}^\nu \partial_{[\mu} N_{\nu]}{}^\rho = 0, \tag{2.26}$$

$$N \cdot \left((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t\right) = 0, \tag{2.27}$$

$$N \cdot \Delta \cdot N^t = \Delta, \tag{2.28}$$

$$N_\kappa{}^\nu N_\lambda{}^\rho N_\mu{}^\sigma \partial_{[\nu}\Delta_{\rho\sigma]} = 0. \tag{2.29}$$

then these conditions are preserved by the Poisson-Lie transformation.

## 3. Elements of Poisson-Lie T-plurality and transformation of boundary conditions

The Poisson-Lie T-plurality was described in many papers (e.g. [2, 3, 11]) and we sketch here only its main features, mainly to set the notation. The tensor field $\mathcal{F}$ on the Lie group $G$ can be written as

$$\mathcal{F}_{\mu\nu} = e_\mu{}^a(g) F_{ab}(g) e_\nu{}^b(g) \tag{3.1}$$

where the vielbeins $e_\mu{}^a(g)$ are components of the right-invariant Maurer-Cartan forms $dgg^{-1}$ and $F_{ab}(g)$ are matrix elements of bilinear nondegenerate form $F(g)$ on $\mathfrak{g}$, the Lie algebra of $G$. The action of the $\sigma$-model then reads

$$S_{F,A}[g] = \int_\Sigma d^2x\, \rho_-(g) \cdot F(g) \cdot \rho_+(g)^t + \int_{\sigma=0} A - \int_{\sigma=\pi} A, \tag{3.2}$$

where the right-invariant vector fields $\rho_\pm(g)$ are given by

$$\rho_\pm(g)^a \equiv (\partial_\pm g g^{-1})^a = \partial_\pm \phi^\mu\, e_\mu{}^a(g), \quad (\partial_\pm g g^{-1}) = \rho_\pm(g) \cdot T = \partial_\pm \phi \cdot e(g) \cdot T, \tag{3.3}$$

$T_a$ are basis elements of the Lie algebra $\mathfrak{g}$ and $A$ is the 1-form introduced in (2.19).

Similarly, the boundary conditions (2.2) may be expressed in terms of the right-invariant fields, as

$$\rho_-(g)|_{\sigma=0,\pi} = \rho_+(g) \cdot R_\rho|_{\sigma=0,\pi}, \tag{3.4}$$

where

$$R_\rho = e^{-1}(g) \cdot R \cdot e(g). \tag{3.5}$$

The $\sigma$-models that are transformable under Poisson-Lie T-duality can be formulated on a Drinfel'd double $D \equiv (G|\tilde{G})$, a Lie group whose Lie algebra $\mathfrak{d}$ admits a decomposition $\mathfrak{d} = \mathfrak{g} \dot{+} \tilde{\mathfrak{g}}$ into a pair of subalgebras maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $\langle\, .\, ,\, .\, \rangle$. The matrices $F_{ab}(g)$ for the dualizable $\sigma$-models are of the form

$$F(g) = (E_0^{-1} + \Pi(g))^{-1}, \quad \Pi(g) = b(g) \cdot a(g)^{-1} = -\Pi(g)^t, \tag{3.6}$$

where $E_0$ is a constant matrix, $\Pi$ defines the Poisson structure on the group $G$, and $a(g), b(g)$ are submatrices of the adjoint representation of $G$ on $\mathfrak{d}$

$$gTg^{-1} \equiv Ad(g) \triangleright T = a^{-1}(g) \cdot T, \quad g\tilde{T}g^{-1} \equiv Ad(g) \triangleright \tilde{T} = b^t(g) \cdot T + a^t(g) \cdot \tilde{T}, \tag{3.7}$$

where $\tilde{T}^a$ are elements of dual basis in the dual algebra $\tilde{\mathfrak{g}}$, i.e., $\langle\, T_a, \tilde{T}^b\, \rangle = \delta_a^b$.

The bulk equations of motion of the dualizable $\sigma$-models can be written as Bianchi identities for the $\tilde{\mathfrak{g}}$-valued fields

$$(\rho_+)_a = -\rho_+(g)^b F(g)_{cb} (a(g)^{-1})_a^c, \qquad (\rho_-)_a = \rho_-(g)^b F(g)_{bc} (a(g)^{-1})_a^c.$$

These fields can be consequently integrated in terms of suitable $\tilde{h} : \Sigma \to \tilde{G}$,

$$\begin{aligned} \tilde{\rho}_+(\tilde{h})_a = (\partial_+ \tilde{h}\, \tilde{h}^{-1})_a &= -\rho_+(g)^b F(g)_{cb} (a(g)^{-1})_a^c, \\ \tilde{\rho}_+(\tilde{h})_a = (\partial_- \tilde{h}\, \tilde{h}^{-1})_a &= \rho_-(g)^b F(g)_{bc} (a(g)^{-1})_a^c. \end{aligned} \tag{3.8}$$

This procedure defines the lift $l : \Sigma \to D$ of the solution $g : \Sigma \to G$ into the Drinfel'd double. As a consequence, the lift satisfies [2],

$$\langle\, \partial_\pm l l^{-1}, \mathcal{E}^\pm\, \rangle = 0, \tag{3.9}$$

where $l = g\tilde{h}$ and $\mathcal{E}^{\pm}$ are two orthogonal subspaces in $\mathfrak{d}$, spanned by $T + E_0 \cdot \tilde{T}$, $T - E_0^t \cdot \tilde{T}$, respectively. On the other hand, starting from a solution $l$ in the Drinfel'd double we find a corresponding solution $g$ by constructing the decomposition $l = g\tilde{h}$.

In general, there are several decompositions (Manin triples) of a Drinfel'd double that enable to transform one $\sigma$-model and its solutions into others. Let $\hat{\mathfrak{g}} \dot{+} \bar{\mathfrak{g}}$ be another decomposition of the Lie algebra $\mathfrak{d}$. The pairs of dual bases of $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}, \bar{\mathfrak{g}}$ are related by the linear transformation

$$\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix}, \tag{3.10}$$

where the duality of both bases requires

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}^{-1} = \begin{pmatrix} s^t & q^t \\ r^t & p^t \end{pmatrix}, \tag{3.11}$$

i.e.,

$$\begin{aligned} p \cdot s^t + q \cdot r^t &= \mathbf{1}, \\ p \cdot q^t + q \cdot p^t &= 0, \\ r \cdot s^t + s \cdot r^t &= 0. \end{aligned} \tag{3.12}$$

The $\sigma$-model obtained by the plurality transformation is then defined analogously to the original one, namely by substituting

$$\widehat{F}(\hat{g}) = (\widehat{E}_0^{-1} + \widehat{\Pi}(\hat{g}))^{-1}, \quad \widehat{\Pi}(\hat{g}) = \widehat{b}(\hat{g}) \cdot \widehat{a}(\hat{g})^{-1} = -\widehat{\Pi}(\hat{g})^t, \tag{3.13}$$

$$\widehat{E}_0 = (p + E_0 \cdot r)^{-1} \cdot (q + E_0 \cdot s) = (s^t \cdot E_0 - q^t) \cdot (p^t - r^t \cdot E_0)^{-1} \tag{3.14}$$

into (3.1), (3.2). Solutions of the two $\sigma$-models are related by two possible decompositions of $l \in D$, namely

$$l = g\tilde{h} = \hat{g}\bar{h}. \tag{3.15}$$

For $p = s = 0,\ q = r = \mathbf{1}$ we get the so-called Poisson-Lie T-duality where $\hat{G} = \tilde{G},\ G' = G,\ \widehat{E}_0 = E_0^{-1}$. If $G$ is non-Abelian and $\tilde{G}$ is Abelian we call it non-Abelian T-duality.

The corresponding transformation of the gluing matrix $R_\rho$ under the Poisson-Lie T-plurality was found in [1] in the form

$$\widehat{R_\rho} = \widehat{F}^t(\hat{g}) \cdot M_-^{-1} \cdot F^{-t}(g) \cdot R_\rho(g) \cdot F(g) \cdot M_+ \cdot \widehat{F}^{-1}(\hat{g}), \tag{3.16}$$

where

$$M_+ \equiv s + E_0{}^{-1} \cdot q, \quad M_- \equiv s - E_0{}^{-t} \cdot q. \tag{3.17}$$

An obvious drawback of the formula (3.16) is that the transformed gluing matrix $\widehat{R_\rho}$ may depend not only on $\hat{g}$ but also on $g$, i.e., after performing the lift into the double $g\tilde{h} = \hat{g}\bar{h}$ it may depend on the new dual group elements $\bar{h} \in \bar{G}$, which contradicts any reasonable geometric interpretation of the transformed boundary conditions. A solution of this problem is that we admit gluing matrices only in the form

$$R_\rho(g) = F^t(g) \cdot C \cdot F^{-1}(g), \tag{3.18}$$

where $C$ is a constant matrix.[4] Then $\widehat{R_\rho}$ depends only on $\hat{g}$.

The condition (2.24) requiring that $R_\rho$ of the form (3.18) preserves the metric then restricts the form of the matrix $C$ by

$$C \cdot (E_0^{-1} + E_0^{-t}) \cdot C^t = (E_0^{-1} + E_0^{-t}). \tag{3.19}$$

It is an easy exercise [1] to show that eq. (3.19) is preserved under the Poisson-Lie transformation (3.16).

## 4. Examples of three-dimensional $\sigma$-models

The conditions (2.24)–(2.29) can be used in the following way. Let us assume that the tensor $\mathcal{F}$ is given. For the given metric $\mathcal{G}$, i.e. symmetric part of $\mathcal{F}$, we find admissible gluing operators $\mathcal{R}$ from eq. (2.24), i.e. operators orthogonal with respect to $\mathcal{G}$. Then the projector $N$ is determined from eqs. (2.25) and the condition of integrability (2.26) is checked. Finally, the 2-form $\Delta$ is obtained from (2.27), (2.28) and we check the condition (2.29), namely, that it is closed on the brane. The same procedure is then repeated for the dual or plural model with $\widehat{F}$ and $\widehat{R_\rho}$ given by (3.13) and (3.16).

As an example we shall investigate the Poisson-Lie transformations of the $\sigma$-models formulated on the Drinfel'd doubles $D \equiv (G|\tilde{G})$, where $G$ is the Lie group corresponding to one of the nine three-dimensional Lie algebras *Bianchi* 1 - *Bianchi* 9 (for notation see e.g. [12]) and $\tilde{G}$ is the Abelian Lie group corresponding to *Bianchi* 1. We shall denote these Drinfel'd doubles $(X|1)$ where $X$ is the number of the Bianchi algebra.

The matrix $\Pi$ vanishes for Abelian $\tilde{G}$ so that $F(g) = E_0$ and

$$\mathcal{F}_{\mu\nu} = e_\mu{}^a(g)(E_0)_{ab}\, e_\nu{}^b(g). \tag{4.1}$$

We choose the constant matrix $E_0$ as

$$E_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.2}$$

so that we work with an indefinite metric on $G$.

Our task is to choose gluing operators $\mathcal{R}$ producing $\Delta$ and $N$ that satisfy the conditions (2.24)–(2.29) and check whether the transformed gluing operators $\widehat{\mathcal{R}}$ which are expressed in the non-coordinate frame of the right-invariant fields by (3.16), produce $\widehat{\Delta}$ and $\widehat{N}$ satisfying the conditions (2.24)–(2.29) if and only if the original ones do.

The generic solution of eq. (2.24) for $E_0$ given by (4.2) is

$$R_\rho = \begin{pmatrix} \beta & \gamma & -\frac{\gamma^2}{2\beta} \\ \frac{(\alpha-\epsilon)\beta}{\gamma} & \alpha & -\frac{(\alpha+\epsilon)\gamma}{2\beta} \\ -\frac{(\alpha-\epsilon)^2\beta}{2\gamma^2} & \frac{1-\alpha^2}{2\gamma} & \frac{(\alpha+\epsilon)^2}{4\beta} \end{pmatrix}, \tag{4.3}$$

[4] In general, one can admit $C$ dependent on some combinations of coordinates of $G$ that transform by Poisson-Lie T-plurality to coordinates on $\widehat{G}$ (see [1]).

where $\epsilon = \pm 1$, and $\alpha, \beta, \gamma$ are real constants such that $\beta, \gamma \neq 0$ .

Note that the conditions (2.25), (2.27), (2.28) can be calculated even in the non-coordinate frame where $\mathcal{F} = E_0$, therefore $N_\rho$ and $\Delta_\rho$ are independent of $G$. Moreover, the condition (2.26) holds for all ranks of $\mathcal{N}$ but two and the condition (2.29) holds for all ranks of $\mathcal{N}$ but three on dimensional grounds.

Solving eq. (2.25) for the above given matrix $R_\rho$ and $\epsilon = 1$ we get the identity projector $\mathcal{N} = id$, and for $\epsilon = -1$ we get $N = e(g) \cdot N_\rho \cdot e(g)^{-1}$ where[5]

$$N_\rho = \begin{pmatrix} \frac{n_1\beta^2}{\alpha\gamma+\gamma} + 1 & \frac{n_2\beta^2}{\alpha\gamma+\gamma} & -\frac{\beta^2\left(n_2\beta(\alpha-2\gamma-1)+2\left(n_1\beta^2+\alpha\gamma+\gamma\right)\right)}{2(\alpha+1)^2\gamma^2} \\ \frac{n_1\beta(\alpha-2\gamma-1)}{2(\alpha+1)\gamma} & \frac{n_2\beta(\alpha-2\gamma-1)}{2(\alpha+1)\gamma} + 1 & -\frac{\beta(\alpha-2\gamma-1)\left(n_2\beta(\alpha-2\gamma-1)+2\left(n_1\beta^2+\alpha\gamma+\gamma\right)\right)}{4(\alpha+1)^2\gamma^2} \\ n_1 & n_2 & \frac{\beta(-\alpha n_2+2\gamma n_2+n_2-2n_1\beta)}{2(\alpha+1)\gamma} \end{pmatrix} \tag{4.4}$$

and $n_1, n_2$ are arbitrary constants. The rank of the latter projector is 2.

For $\epsilon = 1$, the condition (2.26) is satisfied trivially as the distribution of tangent vector spaces of the space filling D-branes is identical with the tangent spaces of the manifold. The conditions (2.27), (2.28) yield

$$\Delta_\rho = \begin{pmatrix} 0 & -\frac{2\gamma}{\alpha+2\beta+1} & \frac{\alpha-2\beta+1}{\alpha+2\beta+1} \\ \frac{2\gamma}{\alpha+2\beta+1} & 0 & -\frac{2(\alpha-1)\beta}{\gamma(\alpha+2\beta+1)} \\ -\frac{\alpha-2\beta+1}{\alpha+2\beta+1} & \frac{2(\alpha-1)\beta}{\gamma(\alpha+2\beta+1)} & 0 \end{pmatrix}, \quad \Delta = e(g) \cdot \Delta_\rho \cdot e(g)^t. \tag{4.5}$$

The form of $e(g)$ and therefore also the condition (2.29) depend on $G$.

The results for $\epsilon = 1$ are:

- For *Bianchi* 1,2,$6_0, 7_0$ the condition (2.29) is satisfied for any gluing matrix of the form (4.3).
- For *Bianchi* 3,4,5,$6_a, 7_a$ the condition (2.29) is satisfied if and only if $\alpha = 1$.

If $\epsilon = -1$, the results are:

- For *Bianchi* 1,5 the condition (2.26) is satisfied for any gluing matrix of the form (4.3).
- For *Bianchi* $3, 6_a$ the condition (2.26) is satisfied if and only if $\beta = -1,\ \gamma = \pm 2$ or $\alpha = \frac{\gamma+2\gamma\beta\pm 2\beta}{\gamma\mp 2\beta}$.
- For *Bianchi* $6_0$ the condition (2.26) is satisfied if and only if $\alpha = 1 + 2\beta \pm 2\gamma$.
- For *Bianchi* $2, 4, 7_0, 7_a$ the condition (2.26) is never satisfied.

It is too complicated to check the conditions (2.26) and (2.29) for the simple groups that correspond to *Bianchi* $8, 9$ and the generic solution of eq. (3.19). Nevertheless, we

[5]This holds for generic values of $\alpha, \beta, \gamma$. Cases $\epsilon = 1,\ \alpha = -1-2\beta$ and $\epsilon = -1,\ \alpha = 1-2\beta\pm 4\sqrt{-\beta},\ \alpha = -1$ when forms of $\mathcal{N}$ are different were analyzed separately and the invariance under T-duality was also confirmed.

can calculate them at least for a particular gluing matrix

$$R_\rho = \begin{pmatrix} 0 & 0 & \frac{1}{\beta} \\ 0 & 1 & -\frac{\alpha}{\beta} \\ \beta & \alpha & -\frac{\alpha^2}{2\beta} \end{pmatrix} \tag{4.6}$$

that is a special solution of eq. (3.19). Solving eq. (2.25) for the above given matrix $R_\rho$ we get the projector

$$N_\rho = \begin{pmatrix} \frac{n}{\beta}+1 & 0 & \frac{n+\beta}{\beta^2} \\ -\frac{n\alpha}{2\beta} & 1 & -\frac{\alpha(n+\beta)}{2\beta^2} \\ -n & 0 & -\frac{n}{\beta} \end{pmatrix}, \tag{4.7}$$

where $n$ is an arbitrary constant. Rank of this projector is 2 so that the condition (2.29) is satisfied trivially and

- For *Bianchi* 8 the condition (2.26) is satisfied if and only if $\alpha = \pm 2\sqrt{\beta^2 - 1}$.
- For *Bianchi* 9 the condition (2.26) is never satisfied.

### 4.1 Non-Abelian T-duality

As a next step, we shall investigate the constraints for the dual gluing matrices obtained by the Poisson-Lie T-duality that interchanges $G$ and $\tilde{G}$. We have proven in [1] that the so-called conformal condition (2.24) is preserved by the transformation (3.16) so it is not necessary to check it. For the models on the Drinfel'd doubles $(X|1)$, the Poisson-Lie T-duality reduces to the non-Abelian T-duality and the gluing matrices of the dual models are

$$\widehat{R_\rho} = -\widehat{F}^t(\hat{g}) \cdot E_0^t \cdot C \cdot E_0^{-1} \cdot \widehat{F}^{-1}(\hat{g}) = -\Big(\mathbf{1} - E_0^{-t} \cdot \hat{\Pi}(\hat{g})\Big)^{-1} \cdot C \cdot \Big(\mathbf{1} + E_0^{-1} \cdot \hat{\Pi}(\hat{g})\Big). \tag{4.8}$$

They depend on the choice of $G$ which determines the matrices $\widehat{\Pi}$. The projectors $\widehat{\mathcal{N}}$ are obtained from (2.25) and it turns out that the rank of the projector $\widehat{\mathcal{N}}$ is independent of $G$. For $\epsilon = 1$ it is equal to 2 while for $\epsilon = -1$ it is equal to 3. It means that for $\epsilon = 1$ the nontrivial condition is (2.26) while for $\epsilon = -1$ it is the condition (2.29).

For the matrix (4.3) and $\epsilon = 1$ we get:

- *Bianchi* 1,2,$6_0, 7_0$: The condition (2.26) for $\widehat{\mathcal{R}}$ is satisfied for any gluing matrix of the form (4.3).
- *Bianchi* 3,4,5,$6_a, 7_a$: The condition (2.26) for $\widehat{\mathcal{R}}$ is satisfied if and only if $\alpha = 1$.

For the matrix (4.3) and $\epsilon = -1$ we get:

- *Bianchi* 1,5: The condition (2.29) for $\widehat{\mathcal{R}}$ is satisfied for any gluing matrix of the form (4.3).
- *Bianchi* $3, 6_a$: The condition (2.29) for $\widehat{\mathcal{R}}$ is satisfied if and only if $\beta = -1,\ \gamma = \pm 2$ or $\alpha = \frac{\gamma + 2\gamma\beta \pm 2\beta}{\gamma \mp 2\beta}$.
- *Bianchi* $6_0$: The condition (2.29) for $\widehat{\mathcal{R}}$ is satisfied if and only if $\alpha = 1 + 2\beta \pm 2\gamma$.

- *Bianchi* $2, 4, 7_0, 7_a$: The condition (2.29) for $\widehat{\mathcal{R}}$ is never satisfied.

For the matrix (4.6) the projectors $\widehat{\mathcal{N}}$ obtained from (2.25) have the rank equal to 3 so that the condition (2.26) is satisfied trivially and for:

- *Bianchi* 8 the condition (2.29) is satisfied if and only if $\alpha = \pm 2\sqrt{\beta^2 - 1}$.

- *Bianchi* 9 the condition (2.29) is never satisfied.

Comparing the above given results with those in the previous subsection we see that the conditions (2.24)–(2.29) are preserved under the non-Abelian T-duality. We have also checked in examples that the conditions are preserved under the Poisson-Lie T-plurality as well.

## 5. Invariance of the constraints for the boundary conditions under the Poisson-Lie T-plurality

As we have noted in section 2, it is not a priori clear what kind of constraints should be imposed on the gluing operator $\mathcal{R}$ so that on one hand it properly defines the boundary conditions as D-branes and on the other hand these constraints are preserved under the Poisson-Lie T-plurality. The examples in the previous section indicate that we may have managed to establish the right set of constraints, namely (2.24)–(2.29). We have shown in [1] that (2.24) is preserved under Poisson-Lie T-plurality. It remains to be shown that the others are invariant under the Poisson-Lie transformations as well.

### 5.1 An alternative formulation of the consistency conditions on the gluing operator

As it is difficult to find the Poisson-Lie transformation of the projector $\mathcal{N}$ it is convenient to reformulate the conditions (2.25)–(2.29) without its explicit use, i.e., using the gluing operator $\mathcal{R}$ only. This will also prove that the conditions (2.25)–(2.29) do not depend on the non-unique choice of the projector $\mathcal{N}$ and that we don't have to impose the condition (2.28).

For this purpose we recall eq. (2.6)

$$\mathrm{Ran}\,\mathcal{N} = \mathrm{Ran}\,(\mathcal{R} + id)$$

which means that any condition of the form

$$\mathcal{A} \circ \mathcal{N} = 0, \quad \text{i.e.,} \quad N \cdot A = 0$$

can be equivalently written as

$$\mathcal{A} \circ (\mathcal{R} + id) = 0, \quad \text{i.e.,} \quad (R + \mathbf{1}) \cdot A = 0.$$

Consequently, the condition (2.27) can be equivalently written as

$$(R + \mathbf{1}) \cdot \left((\mathcal{F} + \Delta) - (\mathcal{F} + \Delta)^t \cdot R^t\right) = 0. \tag{5.1}$$

Similarly, the condition (2.29), which when expressed in the basis-free form reads

$$\mathrm{d}\Delta(\mathcal{N}(X),\mathcal{N}(Y),\mathcal{N}(Z))=0,\ \forall X,Y,Z\in T_g\mathcal{D},$$

can be equivalently written as

$$(R+\mathbf{1})_\kappa{}^\nu(R+\mathbf{1})_\lambda{}^\rho(R+\mathbf{1})_\mu{}^\sigma\partial_{[\nu}\Delta_{\rho\sigma]}=0. \tag{5.2}$$

Besides that, we recall that the condition (2.26) is just a statement that the distribution (of non-constant dimension)

$$\Lambda : g\in G\to \mathrm{Ran}\,(\mathcal{R}+id)|_g\subseteq T_gG$$

is in involution,

$$[\Lambda,\Lambda]\subseteq\Lambda \tag{5.3}$$

and consequently by Frobenius Theorem completely integrable. Such a statement is obviously independent of the particular choice of the projector $\mathcal{N}$ (although it doesn't have the nice form $0=\ldots$ of eq. (2.26)).

Finally, we look for the the 2-form $\Delta$. We notice that by virtue of eq. (2.24) the matrix

$$(R+\mathbf{1})\cdot\left(\mathcal{F}-\mathcal{F}^t\cdot R^t\right)$$

is skew-symmetric and consequently has the form

$$(R+\mathbf{1})\cdot M\cdot(R+\mathbf{1})^t$$

for some antisymmetric matrix $M$ related to $\mathcal{F},R$ (and, in general, non-unique). Therefore, the condition (5.1) takes the form

$$(R+\mathbf{1})\cdot(\Delta+M)\cdot(R+\mathbf{1})^t=0 \tag{5.4}$$

and, when considered as an equation for $\Delta$, has a solution, e.g. $\Delta=-M$.

Moreover, we can show that the condition (5.2) doesn't depend on the particular choice of a solution of the equation (5.4). It suffices to consider

$$\Upsilon=(R+\mathbf{1})\cdot\Delta\cdot(R+\mathbf{1})^t=(R+\mathbf{1})\cdot\left(\mathcal{F}^t\cdot R^t-\mathcal{F}\right)$$

and compute the expression

$$\partial_\vartheta\Upsilon_{[\mu\nu}(R+\mathbf{1})_{\lambda]}{}^\vartheta$$

using the two ways of expressing $\Upsilon$. Due to the integrability condition (5.3) written in terms of generators $(R+\mathbf{1})_\nu{}^\sigma\partial_\sigma$ of the distribution $\Lambda$ there exist functions $\gamma_{\mu\nu}{}^\kappa$ such that

$$\partial_\vartheta(R+\mathbf{1})_{[\nu}{}^\sigma(R+\mathbf{1})_{\mu]}{}^\vartheta=\gamma_{\mu\nu}{}^\kappa(R+\mathbf{1})_\kappa{}^\sigma.$$

Using this fact one finds by comparison of different expressions for $\partial_\vartheta\Upsilon_{[\mu\nu}(R+\mathbf{1})_{\lambda]}{}^\vartheta$ that

$$\begin{aligned}&(R+\mathbf{1})_{[\mu}{}^\rho(R+\mathbf{1})_\nu{}^\sigma(R+\mathbf{1})_{\lambda]}{}^\vartheta\partial_\vartheta\Delta_{\rho\sigma}=\\&\quad\partial_\vartheta\left(\mathcal{F}^t\cdot R^t-\mathcal{F}\right)_{\rho[\nu}(R+\mathbf{1})_\mu{}^\rho(R+\mathbf{1})_{\lambda]}{}^\vartheta-\left(\mathcal{F}^t\cdot R^t-\mathcal{F}\right)_{\rho[\nu}\frac{\partial}{\partial y^\vartheta}(R+\mathbf{1})_\mu{}^\rho(R+\mathbf{1})_{\lambda]}{}^\vartheta\end{aligned}$$

(note that index $\vartheta$ in $\frac{\partial}{\partial y^\vartheta} \equiv \partial_\vartheta$ is not antisymmetrized, the antisymmetrization on the right hand side involves $\mu, \nu, \lambda$ only).

To sum up, we have found that an equivalent formulation of the condition (5.2) which doesn't depend on the particular choice of $\Delta$ exists and has the form

$$\partial_\vartheta \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta - \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} \frac{\partial}{\partial y^\vartheta} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta = 0. \tag{5.5}$$

We mention that although the functions $\gamma_{\mu\nu}{}^\kappa$ do not appear in the final expression their existence was important in intermediate steps, i.e. the conditions (5.2) and (5.5) are equivalent only if the integrability condition (5.3) holds.

Watchful reader may notice that we have not imposed the condition (2.28) yet. This condition restricts the field strength $\Delta$ only to the physically relevant degrees of freedom and it is reasonable to apply it from this viewpoint. On the other hand, it requires the knowledge of the explicit form of the projector $\mathcal{N}$ which we want to avoid. Under the assumption that the conditions (5.3), (5.5) hold we take any projector $\mathcal{N}$ and any $\Delta$ satisfying (5.1) and construct

$$\tilde{\Delta} = N \cdot \Delta \cdot N^t$$

which also satisfies the conditions (5.1), (5.2) and in addition it satisfies the condition (2.28). The influence of $\Delta$ and $\tilde{\Delta}$ on the motion of strings, i.e. extrema of the action (2.18), is exactly the same. Therefore we may consider $\Delta$ and $\tilde{\Delta}$ physically equivalent and forget the condition (2.28) altogether.

In summary we may write all conditions defining a consistent gluing operator $\mathcal{R}$ as

$$R \cdot \mathcal{G} \cdot R^t = \mathcal{G}, \tag{5.6}$$

$$[\Lambda, \Lambda] \subseteq \Lambda, \qquad \Lambda(g) = \mathrm{Ran}(\mathcal{R} + id)|_g, \tag{5.7}$$

$$\begin{aligned} &\partial_\vartheta \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta - \\ &- \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)_{\rho[\nu} \frac{\partial}{\partial y^\vartheta} (R+\mathbf{1})_\mu{}^\rho (R+\mathbf{1})_{\lambda]}{}^\vartheta = 0. \end{aligned} \tag{5.8}$$

Given such an operator $\mathcal{R}$ we can find the field strength $\Delta$ (using eq. (5.1)) and the projector $\mathcal{N}$ such that conditions (2.24)–(2.29) hold. Both $\mathcal{N}$ and $\Delta$ are in general non-unique but lead to the same dynamics of the strings on the classical level, i.e., the extrema of the action (2.18).

### 5.2 Lift of D-branes to the Drinfel'd double

We can define the lift of a D-brane $\mathcal{D} \subset G$ given by (2.3) to the Drinfel'd double as an integral manifold of the distribution generated by

$$\partial_\tau l|_{\sigma=0,\pi} = \partial_- l|_{\sigma=0,\pi} + \partial_+ l|_{\sigma=0,\pi}. \tag{5.9}$$

From $l = g\tilde{h}$, (3.8) and (3.7) we get

$$\begin{aligned} \partial_\tau l\, l^{-1} &= (\rho_-(g) + \rho_+(g)) \cdot T + (\tilde{\rho}_-(\tilde{h}) + \tilde{\rho}_+(\tilde{h})) \cdot Ad(g)(\tilde{T}) \\ &= (\rho_-(g) + \rho_+(g)) \cdot T + (\rho_-(g) \cdot F(g) - \rho_+(g) \cdot F^t(g))(a^{-t}(g) \cdot b^t(g) \cdot T + \tilde{T}) \end{aligned}$$

On the boundary we get from (3.4), (3.6) and (3.18)

$$\begin{aligned}\partial_\tau l\, l^{-1}|_{\sigma=0,\pi} &= \rho_+(g)|_{\sigma=0,\pi}\cdot F^t(g)\cdot \\ &\qquad [(F^{-t}(g)+C\cdot F^{-t}(g))\cdot T+(C-\mathbf{1})\cdot(a^{-t}(g)\cdot b^t(g)\cdot T+\tilde{T})] \\ &= \rho_+(g)|_{\sigma=0,\pi}\cdot F^t(g)\cdot[(E_0^{-t}+C\cdot E_0^{-1})\cdot T+(C-\mathbf{1})\cdot\tilde{T}]\end{aligned} \tag{5.10}$$

As $\rho_+(g)|_{\sigma=0,\pi}$ is arbitrary and $F(g)$ is invertible we see that the vectors tangent to the lifted D-branes pulled to the unit of the Drinfel'd double form the vector subspace $V_{\mathcal{D}}$ of $\mathfrak{d}$

$$V_{\mathcal{D}}=\mathrm{span}(A^{ab}T_b+B^a{}_b\tilde{T}^b), \tag{5.11}$$

where the matrices $A$ and $B$ are

$$A=E_0^{-t}+C\cdot E_0^{-1},\quad B=C-\mathbf{1}. \tag{5.12}$$

This subspace is isotropic because

$$\langle(A\cdot T+B\cdot\tilde{T})^t,A\cdot T+B\cdot\tilde{T}\rangle=C\cdot E_0^{-t}\cdot C^t-E_0^{-t}+C\cdot E_0^{-1}\cdot C^t-E_0^{-1}=0 \tag{5.13}$$

due to (3.19). Moreover one can see that the subspace is maximally isotropic as the block matrix

$$\begin{pmatrix}A\\ B\end{pmatrix}=\begin{pmatrix}E_0^{-t}+C\cdot E_0^{-1}\\ C-\mathbf{1}\end{pmatrix} \tag{5.14}$$

has the same rank as the block matrix

$$\begin{pmatrix}E_0^{-t}+E_0^{-1}\\ C-\mathbf{1}\end{pmatrix}, \tag{5.15}$$

whose rank is dim $\mathfrak{g}$, because $E_0^{-t}+E_0^{-1}=E_0^{-1}\cdot(E_0+E_0^t)\cdot E_0^{-t}=E_0^{-1}\cdot\mathcal{G}(e)\cdot E_0^{-t}$ is an invertible matrix.

The space $V_{\mathcal{D}}$ is invariant under the Poisson-Lie transformation by construction, nevertheless, one may check it directly from the transformation properties of $T,\tilde{T},E_0$ and $C$. We shall show that the condition (5.8) for admissible gluing matrix $R$ is equivalent to a statement that the isotropic subspace $V_{\mathcal{D}}$ is also a subalgebra.

First of all we shall rewrite the matrices occurring in (5.8) in terms of the matrices (5.12) defining the space $V_{\mathcal{D}}$.

$$R+\mathbf{1}=\mathcal{F}^t\cdot(A_c+B_c\cdot\Pi_c),\quad \mathcal{F}^t\cdot R^t-\mathcal{F}=B_c{}^t\cdot\mathcal{F}, \tag{5.16}$$

where

$$A_c=e^{-t}(g)\cdot A\cdot e^{-1}(g),\quad B_c=e^{-t}(g)\cdot B\cdot e^t(g),\quad \Pi_c=e^{-t}(g)\cdot\Pi(g)\cdot e^{-1}(g). \tag{5.17}$$

The condition (5.8) then acquires the form

$$\left[\mathcal{F}^t\cdot(A_c+B_c\cdot\Pi_c)\right]_{[\lambda}{}^{\rho}\left[\mathcal{F}^t\cdot\left(A_c\cdot\partial_\rho B_c{}^t-\partial_\rho A_c\cdot B_c{}^t-B_c\cdot\partial_\rho\Pi_c\cdot B_c{}^t\right)\cdot\mathcal{F}\right]_{\mu\nu]}=0. \tag{5.18}$$

(Many terms occurring during derivation of this expression cancel by total antisymmetrization in $\lambda, \mu, \nu$.) Using (3.1) and the fact that both $e(g)$ and $F(g)$ are invertible we can simplify the above equation to

$$[(A + B\cdot\Pi(g))\cdot e^{-1}(g)]^{[a\rho}\left(\left[2\,(A + B\cdot\Pi(g))\cdot e^{-1}(g)\cdot\partial_\rho e(g) - B\cdot\partial_\rho\Pi(g)\right]\cdot B^t\right)^{bc]} = 0. \tag{5.19}$$

(The antisymmetrization involves only the indices $a, b, c$.) For the derivatives of $e$ we can use Maurer-Cartan equations, and derivatives of $\Pi(g)$ are

$$\partial_\rho\Pi^{ik} = -(a^{-1})_j{}^i\,\tilde{f}^{jm}{}_n\,\overset{\mathrm{L}}{e}_\rho{}^n(a^{-1})_m{}^k, \tag{5.20}$$

where $\overset{\mathrm{L}}{e}_\mu{}^n$ are components of the left-invariant form $\overset{\mathrm{L}}{e}(g) = e(g)\cdot a(g)$. All that gives

$$(A + B\cdot\Pi(g))^{[ai}\left[f_{ij}{}^k(A + B\cdot\Pi(g))^{bj}B^{c]}{}_k + a_i{}^r(g)\tilde{f}_r^{jk}(B\cdot a^{-t}(g))^b{}_j(B\cdot a^{-t}(g))^{c]}{}_k\right] = 0. \tag{5.21}$$

(where we again antisymmetrize in $a, b, c$ only). We define a mixed product on the Drinfel'd double

$$\langle\langle\ X, Y, Z\ \rangle\rangle := \langle\ [X, Y], Z\ \rangle. \tag{5.22}$$

It is totally antisymmetric and Ad-invariant. In terms of this mixed product we can write the above condition as

$$\langle\langle\ (A\cdot T + B\cdot\Pi(g)\cdot T)^{[a}\,,\ (A\cdot T - B\cdot\Pi(g)\cdot T + B\cdot\tilde{T})^b\,,\ (B\cdot\tilde{T})^{c]}\ \rangle\rangle = 0. \tag{5.23}$$

The antisymmetry of the mixed product and antisymmetrization in indices $a, b, c$ imply

$$\langle\langle\ X^{[a}, Y^b, Z^{c]}\ \rangle\rangle = \langle\langle\ X^{[a}, Z^b, Y^{c]}\ \rangle\rangle = \langle\langle\ Z^{[a}, X^b, Y^{c]}\ \rangle\rangle \tag{5.24}$$

that allows to rewrite the left-hand side of (5.23) as

$$\begin{gathered}\langle\langle\ (A\cdot T)^{[a}\,,\ (A\cdot T)^b\,,\ (B\cdot\tilde{T})^{c]}\ \rangle\rangle + \langle\langle\ (A\cdot T)^{[a}\,,\ (B\cdot\tilde{T})^b\,,\ (B\cdot\tilde{T})^{c]}\ \rangle\rangle\\ -\langle\langle(B\cdot\Pi(g)\cdot T)^{[a}\,,\ (B\cdot\Pi(g)\cdot T)^b\,,\ (B\cdot\tilde{T})^{c]}\rangle\rangle + \langle\langle(B\cdot\Pi(g)\cdot T)^{[a}\,,\ (B\cdot\tilde{T})^b\,,\ (B\cdot\tilde{T})^{c]}\rangle\rangle.\end{gathered}$$

The last two terms drop out by isotropy of the subalgebra $\tilde{\mathfrak{g}}$ because they are equal to

$$\begin{aligned}-\frac{1}{3}\langle\langle(B\cdot\Pi(g)\cdot T - B\cdot\tilde{T})^{[a}\,,\ (B\cdot\Pi(g)\cdot T - B\cdot\tilde{T})^b\,,\ (B\cdot\Pi(g)\cdot T - B\cdot\tilde{T})^{c]}\rangle\rangle&\\ = \frac{1}{3}\langle\langle\ (B\cdot a^{-t}(g)\cdot\tilde{T})^{[a}\,,\ (B\cdot a^{-t}(g)\cdot\tilde{T})^b\,,\ (B\cdot a^{-t}(g)\cdot\tilde{T})^{c]}\ \rangle\rangle &= 0.\end{aligned}$$

The first two terms give

$$\frac{1}{3}\langle\langle\ (A\cdot T + B\cdot\tilde{T})^{[a}\,,\ (A\cdot T + B\cdot\tilde{T})^b\,,\ (A\cdot T + B\cdot\tilde{T})^{c]}\ \rangle\rangle = 0 \tag{5.25}$$

and we can drop the antisymmetrization because of antisymmetry of (5.22). Then eq. (5.25) becomes exactly the statement that the maximal isotropic subspace $V_{\mathcal{D}}$ is a subalgebra of the Drinfel'd double, i.e., that for any $v_1, v_2, v_3 \in V_{\mathcal{D}}$ we have

$$\langle\langle\ v_1, v_2, v_3\ \rangle\rangle = 0.$$



JHEP07(2008)122

To sum up, we conclude that the condition (5.8) is in the case of Poisson-Lie dualizable models equivalent to the statement that the maximally isotropic subspace $V_{\mathcal{D}}$ is a subalgebra. Therefore, the condition (5.8) is Poisson-Lie invariant.

We also see that the lifts of D-branes into the Drinfel'd double $D$ acquire the form of cosets $\mathfrak{D}l$ where $\mathfrak{D}$ is the Lie subgroup of $D$ with Lie algebra $V_{\mathcal{D}}$ and $l \in D$. This demonstrates that the gluing matrix formalism naturally leads to D-branes in Drinfel'd double as devised by C. Klimčík and P. Ševera in [8]. Obviously, the D-brane in Drinfel'd double $\mathfrak{D}l$ is an embedded submanifold of $D$ whenever the condition (5.8) is satisfied, irrespective of the condition (5.7). That leads us to a natural hypothesis that in our case of dualizable gluing operators the distribution $\Lambda : g \in G \to \mathrm{Ran}\,(\mathcal{R} + id)|_g$ is integrable by virtue of the condition (5.8) alone.

In order to show that the distribution $\Lambda$ is integrable we define a coset projection map

$$\pi : \ D \to G : \ l = g\tilde{h} \mapsto g.$$

The D-brane in $G$ passing through $g_0$ is then obtained as $\pi(\mathfrak{D}g_0\tilde{h}_0)$ for some $\tilde{h}_0 \in \tilde{G}$ provided that it is well-defined. That it is indeed so can be seen from the fact that for any $l, l' \in D$ such that $\pi(l) = \pi(l')$ we obviously have

$$\pi \circ R_l = \pi \circ R_{l'} \tag{5.26}$$

and consequently for any $\mathfrak{D}l_1, \mathfrak{D}l_2$ such that $\pi(\mathfrak{D}l_1) \cap \pi(\mathfrak{D}l_2) \neq \emptyset$ we find that intersecting D-branes in $G$ coincide, i.e. $\pi(\mathfrak{D}l_1) = \pi(\mathfrak{D}l_2)$, and are submanifolds. Consequently, $\{\pi(\mathfrak{D}l) | l \in D\}$ form a foliation (of non-constant dimension) of the group $G$ and the distribution $\Lambda$ consisting of tangent spaces to this foliation is by definition integrable.

For a more explicit derivation it is sufficient to consider a basis of right-invariant vector fields on $D$ extended from a basis $(e_k(e))$ of $V_{\mathcal{D}}$ by

$$e_k(l) = (R_l)_* e_k(e)$$

and project them by $\pi_*$

$$E_k(g) = \pi_* e_k(g\tilde{h}).$$

Such $E_k$ are well-defined vector fields on $G$, i.e. don't depend on the choice of $\tilde{h}$, due to eq. (5.26), and define the distribution $\Lambda|_g = \mathrm{span}\{E_k(g)\}$ by construction of the lift. Because $e_k$ close under the commutator, also $E_k$ do so due to $\pi_*([e_j, e_k]) = [\pi_*(e_j), \pi_*(e_k)]$ and consequently the distribution $\Lambda$ is integrable.

A further question arises concerning the generality of our description, i.e. whether any D-brane configuration described in the language of [8] can be expressed in terms of gluing matrices. Let us suppose that we are given an arbitrary maximally isotropic subalgebra $V_{\mathcal{D}}$ of the Drinfel'd double algebra $\mathfrak{d}$, i.e.

$$V_{\mathcal{D}} = \mathrm{span}\{K^{ab} T_b + L^a{}_b \tilde{T}^b\}$$

where $K, L$ are arbitrary matrices such that

$$K \cdot L^t + L \cdot K^t = 0$$

and rank $(K, L) = \dim G$. Does a matrix $C$ exist such that there is an equivalent description

$$V_{\mathcal{D}} = \mathrm{span}\{A^{ab}T_b + B^a{}_b\tilde{T}^b\}$$

where

$$A = E_0^{-t} + C \cdot E_0^{-1}, \quad B = C - \mathbf{1} \ ?$$

The answer is positive provided $L - K \cdot E_0^{-1}$ is regular (invertible) matrix. Indeed, we are looking for an invertible matrix $S$ such that $S \cdot L = A, \ S \cdot K = B$. We find

$$S = \left(E_0^{-t} + E_0^{-t}\right) \cdot \left(L - K \cdot E_0^{-1}\right)^{-1},$$

and

$$C = \left(E_0^{-t} + E_0^{-t}\right) \cdot \left(L - K \cdot E_0^{-1}\right)^{-1} \cdot K + \mathbf{1}.$$

Such matrix $C$ satisfies the condition (3.19). The singular case when $C$ doesn't exist and we cannot use the description based on gluing matrices occurs if and only if there is $v \in V_{\mathcal{D}}, \ v \neq 0$ such that $\langle v, \mathcal{E}^- \rangle = 0$, i.e.,

$$v \in V_{\mathcal{D}} \cap \mathcal{E}^+ \neq 0. \tag{5.27}$$

This is rather exceptional since both $V_{\mathcal{D}}$ and $\mathcal{E}^+$ are $(\dim G)$-dimensional subspaces in $(2 \dim G)$-dimensional vector space $\mathfrak{d}$.

## 6. Conclusions

We have revisited the bosonic version of conditions (2.13)–(2.17) formulated in [5] for the gluing matrices defining boundary conditions for open strings. We have investigated them from the point of view of their invariance under the Poisson-Lie transformations defined by the formulas (3.13), (3.14) and (3.16).

We have seen that in order to keep the conditions invariant under the Poisson-Lie transformations, it is necessary to introduce the electromagnetic field $\Delta$ on the D-branes where the boundary conditions are imposed as in [8]. Besides that we have relaxed the condition (2.14) for the so-called Dirichlet projector $\mathcal{Q}$ that projects onto the space normal to the D-brane as it is not invariant under the Poisson-Lie transformations. We suggest that the proper set of constraints for the gluing matrices is (2.24)–(2.29). The invariance of these constraints under the Poisson-Lie transformations was firstly checked in many examples; some of them were presented in section 4. The invariance was proved in section 5.

Of course, one may imagine also other possible generalizations of the conditions (2.13)–(2.17). One possible approach (in supersymmetric setting) appeared in [14] where the condition (2.16) was not strictly enforced whereas the splitting into Dirichlet and Neumann directions due to (2.13)–(2.14) was retained (together with a stringent restriction $R^2 = 1$). However, that paper dealt with Abelian T-duality only. In the context of Poisson-Lie T-duality it seems that the condition (2.16), i.e. (2.24), has a natural geometric interpretation, namely the isotropy of lifted D-branes (5.13), and it was essential in most of our derivations. That's why we consider it indispensable in our setting. The condition (2.26) is an integrability statement, needed for interpretation of D-branes as submanifolds. The conditions (2.27), (2.29) are equivalent to the vanishing of the boundary term in the variation of action (3.2) and as such are also necessary (as long as one keeps the action in the form (3.2)). The condition (2.28) restricts the field strength $\Delta$ to a specific choice from a physically equivalent set — the physics is not at all influenced by it but it is useful for the uniqueness of $\Delta$. To sum up we believe that all the conditions (2.24)–(2.29) should be imposed in Poisson-Lie T-duality context.

To prove the Poisson-Lie invariance of the constraints (2.24)–(2.29) it was necessary to reformulate them to the form (5.6)–(5.8) that does not contain the (non-unique) projector $\mathcal{N}$. In the end it turned out that the constraints for the gluing matrices

$$R_\rho(g) = F^t(g) \cdot C \cdot F^{-1}(g), \tag{6.1}$$

where $C$ is a constant matrix which satisfies

$$C \cdot (E_0^{-1} + E_0^{-t}) \cdot C^t = (E_0^{-1} + E_0^{-t}), \tag{6.2}$$

are equivalent to the condition that the subspace

$$V_{\mathcal{D}} = \mathrm{span}\left( \left(E_0^{-t} + C \cdot E_0^{-1}\right) \cdot T + (C - \mathbf{1}) \cdot \tilde{T} \right) \tag{6.3}$$

is a maximally isotropic subalgebra. This statement is clearly invariant under the Poisson-Lie transformations because the choice of $V_{\mathcal{D}}$ is independent of the decomposition of the Lie algebra of the Drinfel'd double into the sum of the isotropic subalgebras (Manin triple).

On the other hand, if $V_{\mathcal{D}}$ is a maximally isotropic subalgebra and

$$V_{\mathcal{D}} \cap \mathcal{E}^+ = 0$$

then there is a unique matrix $C$ such that $V_{\mathcal{D}}$ can be written in the form (6.3) and the condition (6.2) is satisfied. The gluing matrix (6.1) then satisfies the consistency conditions (5.6)–(5.8) or equivalently (2.24)–(2.29) where the suitable field strength $\Delta$ is found as a solution of

$$(R + \mathbf{1}) \cdot \Delta \cdot (R + \mathbf{1})^t = (R + \mathbf{1}) \cdot \left(\mathcal{F}^t \cdot R^t - \mathcal{F}\right)$$

and the projector $\mathcal{N}$ is defined by eq. (2.5).

This means that we have shown that the current version of the formulation of transformable boundary conditions in terms of gluing matrices is equivalent to the description originally discovered by C. Klimčík and P. Ševera in [8]. Both approaches can be considered complementary. In their original formulation the invariance of the description is clear from its geometric formulation in the Drinfel'd double and also some of the geometric properties of the lifted D-branes are immediately obvious. However, it may be quite tedious to work out the explicit form of the boundary conditions in the $\sigma$-models on the groups $G, \hat{G}$. (e.g. in the original paper [8] only the Poisson-Lie T-duals of free boundary conditions were worked out in any detail. More complicated D-branes in WZW models found in this way were given in [13].) On the other hand, in our approach these are easy to write down but it required some calculation to show that both the original and transformed boundary conditions satisfy the same consistency requirements (5.6)–(5.8).

Finally, we would like to recall that we have expressed the conditions on gluing matrix in a form independent of the projector $\mathcal{N}$, i.e. (5.6)–(5.8), and that this derivation does not depend at all on the particular structure of Poisson-Lie transformable models or on the fact that we consider group targets. We believe that this formulation may be of use also in other investigations of the properties of gluing matrices.


## Acknowledgments

Some of the ideas that lead into this work originated in joint discussion with Cecilia Albertsson for which we are thankful.


## References


[1] C. Albertsson, L. Hlavaty and L. Snobl, *On the Poisson-Lie T-plurality of boundary conditions*, *J. Math. Phys.* **49** (2008) 032301 [arXiv:0706.0820].

[2] C. Klimčík and P. Ševera, *Dual non-Abelian duality and the Drinfeld double*, *Phys. Lett.* **B 351** (1995) 455 [hep-th/9502122].

[3] R. Von Unge, *Poisson-Lie T-plurality*, *JHEP* **07** (2002) 014 [hep-th/0205245].

[4] C. Albertsson, U. Lindström and M. Zabzine, $N=1$ *supersymmetric* $\sigma$*-model with boundaries. I*, *Commun. Math. Phys.* **233** (2003) 403 [hep-th/0111161].

[5] C. Albertsson, U. Lindström and M. Zabzine, $N=1$ *supersymmetric* $\sigma$*-model with boundaries. II*, *Nucl. Phys.* **B 678** (2004) 295 [hep-th/0202069].

[6] C. Albertsson, U. Lindström and M. Zabzine, *T-duality for the* $\sigma$*-model with boundaries*, *JHEP* **12** (2004) 056 [hep-th/0410217].

[7] C. Albertsson and R.A. Reid-Edwards, *Worldsheet boundary conditions in Poisson-Lie T-duality*, *JHEP* **03** (2007) 004 [hep-th/0606024].

[8] C. Klimčík and P. Ševera, *Poisson-Lie T-duality: open Strings and D-branes*, *Phys. Lett.* **B 376** (1996) 82 [hep-th/9512124].

[9] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112].

[10] S. Stanciu, *D-branes in group manifolds*, *JHEP* **01** (2000) 025 [hep-th/9909163].

[11] E. Tyurin and R. von Unge, *Poisson-Lie T-duality: the path-integral derivation*, *Phys. Lett.* **B 382** (1996) 233 [hep-th/9512025].

[12] L. Snobl and L. Hlavaty, *Classification of 6-dimensional real Drinfeld doubles*, *Int. J. Mod. Phys.* **A 17** (2002) 4043 [math.0202210/].

[13] C. Klimčík, *D-branes in the Euclidean* $AdS_3$ *and T-duality*, *Int. J. Theor. Phys.* **46** (2007) 2443 [hep-th/0211177].

[14] A. Sevrin, W. Staessens and A. Wijns, *The world-sheet description of A and B branes revisited*, *JHEP* **11** (2007) 061 [arXiv:0709.3733].